Lecture 38

Chapter 4 : Linear (aka vector) spaces

* Won't be on the final.

Def: $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$ are linearly independent if the only linear relation $c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$ is the trivial one. $(c_1 = \dots = c_k = 0)$

Def: $W \subset \mathbb{R}^n$ is a subspace if

1) $\vec{0} \in W$
2) $\vec{u}, \vec{v} \in W \rightarrow \vec{u} + \vec{v} \in W$
3) $\vec{v} \in W,\ c \in \mathbb{R} \rightarrow c\vec{v} \in W$

Def: A vector space is a set V (whose elements we call vectors) such that:

1) any $\vec{u}, \vec{v} \in W$ can be added to get another vector $\vec{u} + \vec{v} \in V$
2) any vector $\vec{v} \in V$ can be multiplied by a scaler $c \in \mathbb{R}$ to get another vector.
3) "usual" rules for arithmetic apply

   (e.g. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$, $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$, $\exists\ \vec{0} \in \vec{v}$ s.t. $0 + \vec{u} = \vec{u}$, etc.)

$\boxed{\text{Ex}}$ $V = M_{2\times 2} = 4$ dimensional vector space w basis:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\boxed{\text{Ex}}$ $P = \{$ polynomials $p(x) = c_0 + c_1 x + \dots + c_d x^d \}$ has basis $1, x, x^2, x^3, \dots$

so $\dim V = \infty$

$T : P \to P$

- $T[p(x)] = p'(x)$ is a linear transformation.

i.e. $T(c[p(x)]) = [c\,p(x)]' = c[p'(x)] = c\,T[p(x)]$

$T(p+q) = (p+q)' = p' + q' = T(p) + T(q)$

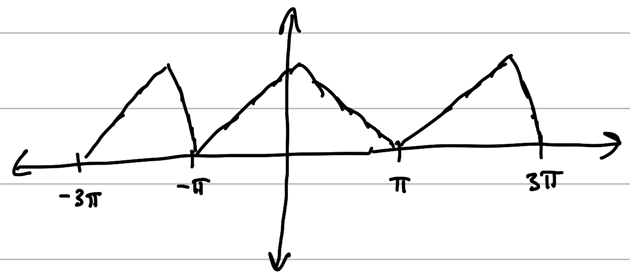
Caution: don't (always) have a replacement for dot product.

$\boxed{\text{Ex}}$ $F = \{$ all functions $f : f : \mathbb{R} \to \mathbb{R} \}$

- has no good notion of orthogonal.

$\boxed{\text{Ex}}$ $C\,[-\pi,\pi] = \{$ continuous 2π - periodic functions $\}$

$= \{$ continuous functions $f : [-\pi,\pi] \to \mathbb{R} \}$



= Subspace of F

Replacement for dot product: given $f, g \in C([-\pi,\pi])$

set

$$\text{"}f \circ g\text{"} = \langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

<u>Fourier's Idea</u>: Sines and Cosines of varying periods give an approximate basis of $C([-\pi,\pi])$.

<u>Obs</u> - $\langle \cos(mx), \cos(nx) \rangle = \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\, dx = 0$ if $m \neq n$
and > 0 if $m = n$

i.e. $\{ \cos(mx), \sin(nx) : m, n \geq 0 \text{ integers} \} = \perp$ set $\to$ linearly independent.

<u>Sadly</u>, these do <u>not</u> span $C([-\pi,\pi])$

<u>However</u>, an $f \in C([-\pi,\pi])$ can be written as an <u>infinite</u> sum

$$f = \sum_{n=0}^{\infty} a_n \cos(nx) + \sum_{m=1}^{\infty} b_m \sin(mx)$$

where $\quad a_n = \dfrac{\langle f, \cos(nx) \rangle}{\langle \cos(nx), \cos(nx) \rangle}$

b_m = similarly (but use sin)